3.1. Să se determine transformata Z a semnalelor:

a) $x[n] = \begin{cases} \left(\frac{1}{3}\right)^n & n \geq 0 \\ \left(\frac{1}{2}\right)^{-n} & n < 0 \end{cases}$

b) $x[n] = \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3} n\right) u[n]$

a) $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} =$$

$$= \sum_{1}^{\infty} \left(\frac{1}{2}\right)^n z^n + \sum_{0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} = \sum_{0}^{\infty} \left(\frac{1}{2}\right)^n z^n - 1 + \sum_{0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} =$$

$$= \sum_{0}^{\infty} \left(\frac{1}{2} z\right)^n + \sum_{0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n - 1 = \frac{1}{1 - \frac{1}{2} z} + \frac{1}{1 - \frac{1}{3} z^{-1}} - 1 =$$

$\sum_{0}^{\infty} \lambda^n = \frac{1}{1-\lambda} \quad |\lambda| < 1$

$\left|\frac{1}{2} z\right| < 1 \quad |z| < 2$

$\left|\frac{1}{3} z^{-1}\right| < 1 \quad |z| > \frac{1}{3}$

$$= \frac{1 - \frac{1}{3} z^{-1} + 1 - \frac{1}{2} z - 1 - \frac{1}{6} + \frac{1}{3} z^{-1} + \frac{1}{2} z}{\left(1 - \frac{1}{2} z\right)\left(1 - \frac{1}{3} z^{-1}\right)}$$

$$= \frac{5/6}{\left(1 - \frac{1}{2} z\right)\left(1 - \frac{1}{3} z^{-1}\right)} \qquad \frac{1}{3} < |z| < 2$$

b) $$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3} n\right) z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{e^{j\frac{\pi}{3} n} - e^{-j\frac{\pi}{3} n}}{2j} z^{-n} =$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left[\left(\frac{1}{2} e^{j\frac{\pi}{3}}\right)^n z^{-n} - \left(\frac{1}{2} e^{-j\frac{\pi}{3}}\right)^n z^{-n}\right] = \frac{1}{2j} \left[\frac{1}{1 - \frac{1}{2} e^{j\frac{\pi}{3}} z^{-1}} - \frac{1}{1 - \frac{1}{2} e^{-j\frac{\pi}{3}} z^{-1}}\right]$$

$$= \frac{1}{2j} \frac{\frac{1}{2} e^{j\frac{\pi}{3}} z^{-1} - \frac{1}{2} e^{-j\frac{\pi}{3}} z^{-1}}{\left(1 - \frac{1}{2} e^{j\frac{\pi}{3}} z^{-1}\right)\left(1 - \frac{1}{2} e^{-j\frac{\pi}{3}} z^{-1}\right)} = \frac{\frac{1}{2} \sin\frac{\pi}{3} z^{-1}}{\left(1 - \frac{1}{2} e^{j\frac{\pi}{3}} z^{-1}\right)\left(1 - \frac{1}{2} e^{-j\frac{\pi}{3}} z^{-1}\right)}$$

3.2. Să se determine toate semnalele posibile $x[n]$ care pot avea transformata Z

$$X(z) = \frac{5}{(1-2z^{-1})(3-z^{-1})^2}$$

$$X(z) = \frac{5}{9(1-2z^{-1})(1-\frac{1}{3}z^{-1})^2} = \frac{5z^3}{9(z-2)(z-\frac{1}{3})^2}$$

$$\frac{X(z)}{z} = \frac{5z^2}{9(z-2)(z-\frac{1}{3})^2} = \frac{1}{9}\left(\frac{A_1}{z-2} + \frac{A_2}{z-\frac{1}{3}} + \frac{A_3}{(z-\frac{1}{3})^2}\right)$$

$$A_1 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} = \frac{5z^2}{(z-\frac{1}{3})^2}\bigg|_{z=2} = \frac{5\cdot 4}{\frac{25}{9}} = \frac{36}{5}$$

$$A_{ik} = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left\{(z-p_k)^m \frac{X(z)}{z}\right\}_{z=p_k}$$

$$A_2 = \frac{d}{dz}\left[\cancel{\left(z-\frac{1}{3}\right)^2}\frac{5z^2}{(z-2)\cancel{(z-\frac{1}{3})^2}}\right]_{z=\frac{1}{3}} = \frac{5\{2z(z-2)-z^2\}}{(z-2)^2}\bigg|_{z=\frac{1}{3}} =$$

$$= \frac{5\{z^2-4z\}}{(z-2)^2}\bigg|_{z=\frac{1}{3}} = -\frac{11}{5} \quad ; \quad A_3 = \frac{5z^2}{z-2}\bigg|_{z=\frac{1}{3}} = -\frac{1}{3}$$

$$\frac{X(z)}{z} = \frac{1}{9}\left[\frac{36/5}{z-2} + \frac{-11/5}{z-\frac{1}{3}} + \frac{-1/3}{(z-\frac{1}{3})^2}\right]; \; X(z) = \frac{1}{9}\left\{\frac{36}{5}\frac{z}{z-2} - \frac{11}{5}\frac{z}{z-\frac{1}{3}} - \right.$$

$$\left. - \frac{1}{3}\frac{z}{(z-\frac{1}{3})^2}\right\} = \frac{1}{9}\left[\frac{36/5}{1-2z^{-1}} - \frac{11}{5}\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{3}\frac{z^{-1}}{(1-\frac{1}{3}z^{-1})^2}\right]$$

pt. $|z| < \frac{1}{3}$ $\quad x[n] = -\frac{1}{9}\left(\frac{36}{5}\right)2^n u[-n-1] + \frac{11}{5}\left(\frac{1}{3}\right)^n u[-n-1] + n\left(\frac{1}{3}\right)^n u[-n-1]$

sistem instabil, necauzal

$\frac{1}{3} < |z| < 2$ $\quad x[n] = -\frac{1}{9}\left(\frac{36}{5}\right)2^n u[-n-1] - \frac{11}{5}\left(\frac{1}{3}\right)^n u[n] - n\left(\frac{1}{3}\right)^n u[n]$

sistem stabil, necauzal

$|z| > 2$ $\quad x[n] = \frac{1}{9}\left(\frac{36}{5}\right)2^n u[n] - \frac{11}{5}\left(\frac{1}{3}\right)^n u[n] - n\left(\frac{1}{3}\right)^n u[n]$

sistem instabil, cauzal

3.3. Fie sistemul descris de următoarea ecuație cu diferențe:

$$y[n] = -0.1\, y[n-1] + 0.2\, y[n-2] + x[n] + x[n-1]$$

a). să se calculeze răspunsul la impuls al sistemului;

b) să se calculeze răspunsul la treapta unitate;

c) să se calculeze răspunsul sistemului la semnalul

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

a)
$$H(z) = \frac{1+z^{-1}}{1+0{,}1z^{-1}-0{,}2z^{-2}} = \frac{1+z^{-1}}{(1+0{,}5z^{-1})(1-0{,}4z^{-1})} =$$

$$= \frac{A_1}{1+0{,}5z^{-1}} + \frac{A_2}{1-0{,}4z^{-1}} \quad ; \quad A_1 = \left.\frac{1+z^{-1}}{1-0{,}4z^{-1}}\right|_{z=-0{,}5} = -\frac{5}{9}$$

$$A_2 = \left.\frac{1+z^{-1}}{1+0{,}5z^{-1}}\right|_{z=0{,}4} = \frac{1+\frac{1}{0{,}4}}{1+0{,}5\frac{1}{0{,}4}} = \frac{14}{9}$$

$$h[n] = \frac{1}{9}\left[14\,(0{,}4)^n - 5(-0{,}5)^n\right]u[n]$$

b)
$$U(z) = \frac{1}{1-z^{-1}} \longleftrightarrow u[n] \qquad Y(z) = \frac{1+z^{-1}}{(1+0{,}5z^{-1})(1-0{,}4z^{-1})}\cdot\frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{B_1}{1+0{,}5z^{-1}} + \frac{B_2}{1-0{,}4z^{-1}} + \frac{B_3}{1-z^{-1}} \quad ; \quad B_1 = -\frac{5}{27} \; ; \; B_2 = -\frac{28}{27} \; ; \; B_3 = \frac{20}{9}$$

$$y[n] = \frac{20}{9}u[n] - \frac{28}{27}(0{,}4)^n u[n] - \frac{5}{27}(-0{,}5)^n u[n]$$

c)
$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$Y(z) = \frac{1+z^{-1}}{(1+0{,}5z^{-1})(1-0{,}4z^{-1})}\cdot\frac{1}{1-\frac{1}{3}z^{-1}} =$$

$$= \frac{C_1}{1+0{,}5z^{-1}} + \frac{C_2}{1-0{,}4z^{-1}} + \frac{C_3}{1-\frac{1}{3}z^{-1}} \quad ; \quad C_1 = -\frac{1}{3} \; ; \; C_2 = \frac{28}{3} \; ; \; C_3 = -8$$

$$y\{n\} = \left[\frac{28}{3}(0,4)^n - \frac{1}{3}(-0,5)^n - 8\left(\frac{1}{3}\right)^n\right] u\{n\}$$

3.4. Să se determine răspunsul la semnalul de intrare $x\{n\} = \left(\frac{1}{3}\right)^n u\{n\}$ al sistemului descris de ecuația cu diferențe

$$y\{n\} = \frac{3}{4}y\{n-1\} - \frac{1}{8}y\{n-2\} + x\{n\}$$

în următoarele condiții inițiale:

a) $y\{-1\} = y\{-2\} = 0$

b) $y\{-1\} = 1$; $y\{-2\} = 1$

a)

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \qquad A(z) = 1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}$$

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \qquad Y_{zs}(z) = H(z)X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$= \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{4}z^{-1}} + \frac{A_3}{1-\frac{1}{3}z^{-1}} \ ; \quad A_1 = 6 \ ; \ A_2 = 3 \ ; \ A_3 = -8$$

$y_{zs}\{n\} = \left[6\left(\frac{1}{2}\right)^n + 3\left(\frac{1}{4}\right)^n - 8\left(\frac{1}{3}\right)^n\right] u\{n\}$ răspunsul de stare zero

b) pentru condițiile inițiale nenule în transformata Z apare componenta suplimentară $Y_{zi}(z) = \frac{N_0(z)}{A(z)}$ unde

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y\{-n\} z^n$$

$$N_0(z) = -\sum_{k=1}^{2} a_k z^{-k} \sum_{n=1}^{k} y\{-n\} z^n = -a_1 z^{-1} y\{-1\} z - a_2 z^{-2}\left(y\{-1\}z + y\{-2\}z^2\right)$$

$$= -a_1 y\{-1\} - a_2 y\{-1\} z^{-1} - a_2 y\{-2\} = -a_1 y\{-1\} - a_2 y\{-2\} - a_2 y\{-1\} z^{-1}$$

$$= \frac{3}{4} - \frac{1}{8} - \frac{1}{8}z^{-1} = \frac{5}{8} - \frac{1}{8}z^{-1}$$

$$Y_{zi}(z) = \frac{5/8 - 1/8\, z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{B_1}{1-\frac{1}{2}z^{-1}} + \frac{B_2}{1-\frac{1}{4}z^{-1}} \ , \quad B_1 = \frac{3}{4} \ ; \ B_2 = -\frac{1}{8}$$

$$y_{zi}\{n\} = \left[\frac{3}{4}\left(\frac{1}{2}\right)^n - \frac{1}{8}\left(\frac{1}{4}\right)^n\right] u\{n\} \qquad y\{n\} = y_{zs}\{n\} + y_{zi}\{n\}$$

Să se determine răspunsul sistemului descris de ecuația cu diferențe:

$$y[n] = 4y[n-1] - 4y[n-2] + x[n] - x[n-1]$$

la semnalul de intrare

$$x[n] = (-1)^n u[n]$$

în cazul în care $y[-1] = 1$ și $y[-2] = -1$ folosind

a) soluția ecuației liniare cu diferențe;

b) transformata Z;

Soluție

a) 1° $y[n] = y_h[n] + y_p[n]$

sau

2° $y[n] = y_{zi}[n] + y_{zs}[n]$

• Ecuația cu diferențe omogenă $y[n] - 4y[n-1] + 4y[n-2] = 0$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_{1,2} = 2$

Rădăcină multiplă $\Rightarrow y_h[n] = C_1 2^n + C_2 n 2^n$

• Soluția particulară a ecuației cu diferențe: $y_p[n] = k(-1)^n u[n]$

Deoarece $y_p[n]$ este o soluție a ecuației cu diferențe:

$$y_p[n] = 4y_p[n-1] - 4y_p[n-2] + x[n] - x[n-1]$$

înlocuind:

$$k(-1)^n u[n] = 4k(-1)^{n-1} u[n-1] - 4k(-1)^{n-2} u[n-2] + (-1)^n u[n] - (-1)^{n-1} u[n-1]$$

pt. $n = 2 \Rightarrow k = -4k - 4k + 1 + 1 \Rightarrow 9k = 2 \Rightarrow k = \frac{2}{9}$

$$y_p = \frac{2}{9}(-1)^n u[n]$$

iar soluția ~~gen~~ totală este:

$$y[n] = \left[C_1 2^n + n C_2 2^n + \frac{2}{9}(-1)^n \right] u[n]$$

1° Pt. n dat avem soluția totală și ecuația cu diferențe:

$$\begin{cases} n=0 & y[0] = C_1 + \frac{2}{9} \quad \text{și} \quad y[0] = 4y[-1] - 4y[-2] + (-1)^0 = 9 \\ n=1 & y[1] = 2C_1 + 2C_2 - \frac{2}{9} \quad \text{și} \quad y[1] = 4y[0] - 4y[-1] + x[1] - x[0] = 30 \end{cases}$$

$$\Rightarrow \cancel{y[0]=} \; C_1 = y[0] - \frac{2}{9} = 9 - \frac{2}{9} = \frac{79}{9} \; ; \quad C_2 = y[1]/2 - C_1 + \frac{1}{9} = \frac{19}{3}$$

$$\Rightarrow y[n] = \left[\frac{79}{9} 2^n + \frac{19}{3} n 2^n + \frac{2}{9}(-1)^n\right] u[n]$$

2° $y[n] = y_{zi}[n] + y_{zs}[n]$

$\underbrace{y_{zi}[n]}$ este răspunsul sistemului la intrare zero $x[n]=0$

Soluția este $y_{zi}[n] = \left[C_1' 2^n + n C_2' 2^n\right] u[n]$

unde $y_{zi}[n] = 4 y_{zi}[n-1] - 4 y_{zi}[n-2]$

$$\begin{cases} n=0 & y_{zi}[0] = C_1' \quad \text{și} \quad y_{zi}[0] = 4y[-1] - 4y[-2] = 8 \\ n=1 & y_{zi}[1] = 2C_1' + 2C_2' \quad \text{și} \quad y_{zi}[1] = 4y_{zi}[0] - 4y_{zi}[-1] = 4\cdot 8 - 4 = 28 \end{cases}$$

$$\Rightarrow C_1' = 8 \quad \text{și} \quad C_2' = 6 \quad \Rightarrow y_{zi}[n] = \left[8 \, 2^n + 6n \, 2^n\right] u[n]$$

$\underbrace{y_{zs}[n]}$ este răspunsul sistemului la stare zero $y[-1] = y[-2] = 0$

$$y_{zs}[n] = \left[C_1'' 2^n + n C_2'' 2^n + \frac{2}{9}(-1)^n\right] u[n]$$

unde $y_{zs}[n] = 4 y_{zs}[n-1] - 4 y_{zs}[n-2] + x[n] - x[n-1]$

$$\begin{cases} n=0 & y_{zs}[0] = C_1'' + \frac{2}{9} \quad \text{și} \quad y_{zs}[0] = (-1)^0 = 1 \\ n=1 & y_{zs}[1] = 2C_1'' + 2C_2'' - \frac{2}{9} \quad \text{și} \quad y_{zs}[1] = 4y_{zs}[0] + x[1] - x[0] = 2 \end{cases}$$

$$\Rightarrow C_1'' = \frac{7}{9} \quad \text{și} \quad C_2'' = y_{zs}[1]/2 - C_1'' + \frac{1}{9} = \frac{1}{3}$$

$$\Rightarrow y_{zs}[n] = \left[\frac{7}{9} 2^n + \frac{1}{3} n 2^n + \frac{2}{9}(-1)^n\right] u[n] \qquad \text{iar } y[n] = y_{zi}[n] + y_{zs}[n]$$

b)

$$Y(z) = H(z)\cdot X(z) + \frac{N_0(z)}{A(z)} \quad \text{unde} \quad \begin{cases} H(z)X(z) \longrightarrow y_{zs}\{n\} \\ \frac{N_0(z)}{A(z)} \xrightarrow{Z^{-1}\{\}} y_{zi}\{n\} \end{cases}$$

$$Y(z) = Y_{zs}(z) + Y_{zi}(z)$$

$$H(z) = \frac{1-z^{-1}}{1-4z^{-1}+4z^{-2}} = \frac{1-z^{-1}}{(1-2z^{-1})^2} \qquad X(z) = \frac{1}{1+z^{-1}}$$

$$A(z) = 1-4z^{-1}+4z^{-2};$$

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y\{-n\} z^{n} = -a_1 z^{-1} y\{-1\} z - a_2 z^{-2}\left(y\{-1\}z + y\{-2\}z^2\right) =$$

$$= -a_1 y\{-1\} - a_2 y\{-2\} - a_2 y\{-1\} z^{-1} \quad \text{unde } a_1 = -4;\ a_2 = 4$$

$$N_0(z) = 8 - 4z^{-1}$$

→ Determinăm originalul lui $Y_{zs}(z)$

$$Y_{zs}(z) = \frac{1-z^{-1}}{1-4z^{-1}+4z^{-2}}\cdot\frac{1}{1+z^{-1}}; \quad Y_{zs}(z) = \frac{z^2(z-1)}{(z-2)^2(z+1)}$$

$$\frac{Y_{zs}(z)}{z} = \frac{z(z-1)}{(z-2)^2(z+1)} = \frac{c_1}{z-2} + \frac{c_2}{(z-2)^2} + \frac{c_3}{z+1}$$

$$c_1 = \frac{d}{dz}\left[\frac{z(z-1)}{z+1}\right]\Bigg|_{z=2} = \frac{(2z-1)(z+1)-(z^2-z)}{(z+1)^2}\Bigg|_{z=2} = \frac{z^2+2z-1}{(z+1)^2}\Bigg|_{z=2} = \frac{7}{9}$$

$$c_2 = \frac{z(z-1)}{z+1}\Bigg|_{z=2} = \frac{2}{3}; \quad c_3 = \frac{z(z-1)}{(z-2)^2}\Bigg|_{z=-1} = \frac{2}{9}$$

$$Y_{zs}(z) = \frac{7}{9}\frac{z}{z-2} + \frac{1}{3}\frac{2\cdot z}{(z-2)^2} + \frac{2}{9}\frac{z}{z+1} = \frac{7}{9}\frac{1}{1-2z^{-1}} + \frac{1}{3}\frac{2\cdot z^{-1}}{(1-2z^{-1})^2} + \frac{2}{9}\frac{1}{1+z^{-1}}$$

$$y_{zs}\{n\} = \left\{\frac{7}{9}\cdot 2^n + \frac{1}{3} n\cdot 2^n + \frac{2}{9}(-1)^n\right\} u\{n\}$$

$$Y_{zi}[z] = \frac{N_0(z)}{A(z)} = \frac{8-4z^{-1}}{1-4z^{-1}+4z^{-2}} = 4\,\frac{2-z^{-1}}{(1-2z^{-1})^2} = 4\,\frac{z(2z-1)}{(z-2)^2}$$

$$\frac{Y_{zi}[z)}{z} = 4\,\frac{2z-1}{(z-2)^2} = 4\left[\frac{c_1'}{z-2} + \frac{c_2'}{(z-2)^2}\right]$$

$$c_1' = \frac{d}{dz}\left[2z-1\right]\Big|_{z=2} = 2 \;;\; c_2' = (2z-1)\Big|_{z=2} = 3$$

$$Y_{zi}(z) = 8\,\frac{z}{z-2} + 6\,\frac{2z}{(z-2)^2} = 8\,\frac{1}{1-2z^{-1}} + 6\,\frac{2z^{-1}}{(1-2z^{-1})^2}$$

$$y_{zi}[n] = \left[8\cdot 2^n + 6\cdot n\cdot 2^n\right] u[n]$$

$$y[n] = y_{zs}[n) + y_{zi}[n) = \left[\left(8+\frac{7}{9}\right)2^n + \left(6+\frac{1}{3}\right)n\,2^n + \frac{2}{9}(-1)^n\right] u[n] =$$

$$= \left[\frac{79}{9}\,2^n + \frac{19}{3}\,n\,2^n + \frac{2}{9}(-1)^n\right] u[n]$$