Time Difference of Arrival Based Source Localization within a Sparse Representation Framework

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***Abstract* — The problem addressed is source localization from time differences of arrival (TDOA). This problem is also referred to as hyperbolic localization and it is non-convex in general. Traditional solutions proposed in the literature have generally poor robustness to errors in the TDOA estimates. More recent methods, which relax the non-convex problem to a convex optimization by applying a semi-definite relaxation (SDR) method, were found to be more robust to TDOA errors than the traditional methods. However, the SDR methods are not optimal in general. In this paper, three convex optimization methods with different computational costs are proposed to improve the hyperbolic localization accuracy. The first method takes an SDR approach to relax the hyperbolic localization to a convex optimization. The second method follows a linearized formulation of the problem and seeks for a biased estimate of improved accuracy. The first two methods perform comparably when the source is inside the convex hull of the sensors. When the source is located outside, the second approach performs better, at the cost of higher computation. A third method is proposed by exploiting the source sparsity. With this, the hyperbolic localization is formulated as an ℓ_1-regularization problem, where the ℓ_1-norm is used as source sparsity constraint. Computer simulations show that the ℓ_1-regularization can offer further improved accuracy, but at the cost of additional computational effort.**

***Index Terms* — Hyperbolic localization, time-difference-of-arrival, sparse representation, ℓ_1-regularization.**

I. INTRODUCTION

Accurate localization of a signal source is a problem of interest in various applications, e.g., [1]. The current work considers source localization in a plane, by sensors that are distributed arbitrarily over the plane. Typically the source location is estimated in two stages. During the first stage, a measure of the received signal, usually the propagation time delay, is estimated at each sensor. In the second stage, the actual location is computed from the time delay estimates.

Time delay estimation (TDE) becomes challenging in multipath propagation environments, where the line of sight (LOS) signal component becomes obscured by multipath reflections. Hence, accurate localization requires techniques capable of resolving the LOS signal component. When the transmitted signal and its transmission time are known at a sensor, the time of arrival (TOA) can be estimated by a variety of techniques. Conventional methods estimate the TOA from the timing of the peaks of the cross-correlation (CC) between the transmitted and received signals, [2], or of a temporal pseudo-spectrum computed from the transmitted and received signals by super-resolution methods, such as MUSIC, [3]. Recently, some potentially even higher resolution estimation techniques have been proposed, based on the observation that propagation channels associated with multipath environments often tend to exhibit a sparse structure in the time domain, i.e., the number of multipaths is much smaller than the number of samples of the received signal. This sparsity has been exploited in TOA estimation and other TOA-related applications, [4, 5]. TOA estimation requires the transmitted signal to be known to the sensors. In many applications, the source may be non-cooperative or otherwise the signal and timing information may not be available at the receiving sensors. The common approach for such a case is to use one of the sensors as reference and measure the time-difference-of-arrival (TDOA) at each of the other sensors with respect to the chosen reference sensor. A method for high resolution TDOA estimation for narrowband sparse multipath channels was developed in [6].

The focus of the current paper is on the *second stage* of non-cooperative sources localization. For any pair of sensors, given their locations, the TDOA estimated at the first stage localizes the source on a hyperboloid with constant range difference between the two sensors. Since the source can occupy only a single point on the hyperbolic curve, TDOA measurements from the other sensors are used to resolve the location ambiguity. The process of finding a solution of the intersection of the hyperbolic curves is referred to as *hyperbolic localization* and is equivalent to solving a system of non-linear equations, [7].

In the literature, there are mainly two traditional approaches to solve the hyperbolic localization problem. The first approach is based on the *nonlinear least squares* (NLS) framework [8] and implies finding the global minimum of a NLS objective function. Under the standard assumption that the TDOA estimates have Gaussian distribution, the global minimum of the objective function corresponds to a maximum likelihood (ML) location estimate, enjoying asymptotic optimality properties, [9]. Although optimum estimation performance can be attained, the algorithm converges to the correct solution only if it is initialized sufficiently close to the final solution. Otherwise, the estimate may be a local minimum, since the objective function may have multimodal features, i.e., the problem is non-convex. This is illustrated in Fig. 1, where one realization of the multimodal objective function is shown for a case with 4 sensors. A second traditional approach is to transform the set of nonlinear equations into a set of linear equations by squaring them and introducing an intermediate variable, expressed as function of the source location, [7, 10-12]. A representative example of this approach is the two-step *weighted least squares* (WLS) method proposed in [7]. This method provides an approximation of the ML estimator for source location. However this approximation holds only when the estimation errors are small, [13].

A third, more recent approach to hyperbolic localization is to relax the non-convex problem to a convex one that can be efficiently solved by standard algorithms, [14]. This can be achieved by applying a *semi-definite relaxation* (SDR) method, [15]. While this approach doesn't guarantee optimality, the solution is generally close enough to the optimal, to at least serve as initialization for a gradient algorithm solving. Moreover, the SDR approach has been found to be more robust to TDOA estimation errors than traditional approaches. In the literature, various SDR methods, each with its own advantages and drawbacks, were proposed to solve different variations of the hyperbolic localization problem, [9, 13, 16, 17].

In the present work, three different methods are proposed to solve the nonlinear system of equations defining the hyperbolic localization problem. The proposed methods improve over existing methods in different scenarios, with different computational costs. The first method is an alternative to the WLS solution by formulating the hyperbolic localization problem as a constrained minimization and relaxing the quadratic relation between the intermediate variable(s) introduced and the source location. The second method is to seek a biased estimate instead of the conventional unbiased estimate produced by the WLS method. This method is developed in a more general biased estimation context discussed in [18] and is also formulated as a constrained minimization problem. Finally, the third method is to introduce a grid over the surveillance area and formulate an objective function related to the likelihood of the source to occupy a certain point on the grid. Exploiting the sparsity of the sources, the problem is formulated as an ℓ_1-regularization, solvable by standard convex optimization algorithms, [14].

The remainder of this paper is organized as follows. Sec. II introduces the basic system model. In Sec. III – V, the three proposed methods for hyperbolic localization are presented. In Sec. VI, numerical simulations are conducted to compare the performance of the proposed methods. Conclusions are listed in Sec. VII.

II. System Model

With the hyperbolic localization problem, the unknown location, $\xi_0 = [x_0, y_0]^{\mathrm{T}}$, of a signal source has to be estimated based on $M-1$ TDOAs estimated by a number M of sensors, $M \geq 3$. The sensors are assumed dispersed within a surveillance area, at arbitrary but precisely known locations, $\xi_k = [x_k, y_k]^{\mathrm{T}}$. Perfect time synchronization is assumed across sensors. One of the sensors, say the first, is used as reference. The estimates express the TDOA with respect to the reference sensor. It is further assumed that the TDOA estimates, τ_{k1}, are available at a fusion center, where the location estimation is performed.

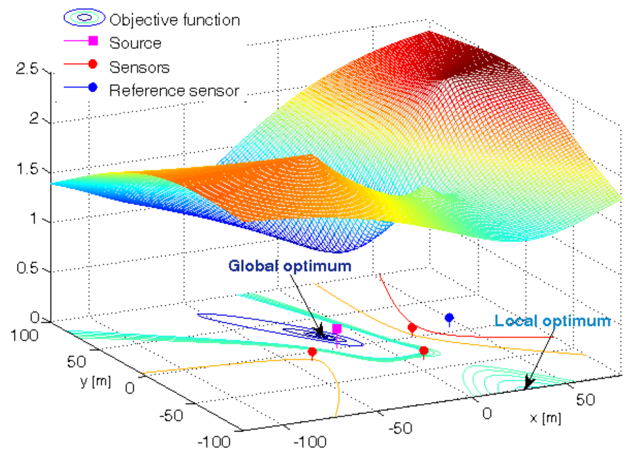

Fig. 1. Non-convex realization of the localization objective function.

The location of the source is estimated by converting the TDOA estimates into range differences, i.e., $d_{k1} = c\tau_{k1}$, for $k = 2, \dots, M$, where c is the speed of light. Denoting the true distance (noise free) value of d by d^{g}, the range differences are commonly modeled, [7],

$$d_{k1} = d_{k1}^{\mathrm{g}}(\xi_0) + n_{k1}, \text{ for } k = 2, \dots, M, \tag{1}$$

where $d_{k1}^{\mathrm{g}} = \|\xi_k - \xi_0\|_2 - \|\xi_1 - \xi_0\|_2$, with $\|\xi_k - \xi_0\|_2 = \sqrt{(x_k - x_0)^2 + (y_k - y_0)^2}$ denoting the Euclidean distance between the source and sensor k. The noise term n_{k1} is usually modeled as a zero mean Gaussian random process. The covariance of $\boldsymbol{n} = [n_{k2}, \dots, n_{kM}]^{\mathrm{T}}$ is denoted by $\boldsymbol{Q}_n = \mathbb{E}\{\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}\}$, where $\mathbb{E}$ is the expectation operator; $\boldsymbol{Q}_n$ is assumed known up to a scalar. Note that because of the common reference, in reality matrix $\boldsymbol{Q}_n$ is not a diagonal matrix. However, it is a common practice in the literature to model the range difference estimation errors as independent across sensor pairs and thus assume $\boldsymbol{Q}_n = \sigma_n^2 \boldsymbol{I}_{M-1}$, where σ_n^2 is the range difference variance of any pair of sensors and $\boldsymbol{I}_{M-1}$ denotes the unity matrix of dimensions $(M-1) \times (M-1)$.

For a number M of sensors, a set of $M(M-1)/2$ TDOA estimated values can be obtained, referred to as the full TDOA set. Instead, by using only one sensor as reference, a set of $M-1$ TDOA estimates is obtained, referred here as the non-redundant TDOA set. It was shown in [19] that if the reference sensor is properly chosen, the non-redundant TDOA set can result in the same localization accuracy as the full set. A procedure for properly choosing the reference sensor can be developed based on the Cramer-Rao lower bound expression, [16, 19]. In the present work, the proper choice of the reference sensor is assumed.

III. An SDR Method for Hyperbolic Localization

In this section, an SDR approach is proposed to solve for the source location ξ_0 estimation from the system of non-linear equations (1). First, both sides of equality (1) are squared and the resulting terms rearranged. By introducing three intermediate variables,

$$\rho = \|\xi_1 - \xi_0\|_2, \ \nu = \rho^2, \ \text{and} \ \gamma = \|\xi_0\|_2^2 , \tag{2}$$

and denoting the noise term $e_k = n_{k1}(2\|\xi_k - \xi_0\|_2 + n_{k1})$, the following linear equations are obtained for $k = 2, \dots, M$:

$$(d_{k1}^2 + 2d_{k1}\rho + \nu) - (\|\xi_k\|_2^2 - 2\xi_k^{\mathrm{T}}\xi_0 + \gamma) = e_k\,. \quad (3)$$

By denoting the left hand side of (3) as $\Lambda_k(\xi_0)$, the dependence on ρ, ν, and γ being implicit, and letting $\boldsymbol{\Lambda}(\xi_0) = [\Lambda_2(\xi_0), \dots, \Lambda_M(\xi_0)]^{\mathrm{T}}$, $\boldsymbol{d} = [d_2, \dots, d_M]^{\mathrm{T}}$, and $\boldsymbol{\xi} = [\xi_2, \dots, \xi_M]^{\mathrm{T}}$, it can be verified that

$$\boldsymbol{\Lambda}(\xi_0) = \mathrm{trace}\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{d} - \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi} + (\nu - \gamma)\boldsymbol{I}_{M-1}\} + 2(\rho\boldsymbol{d} + \xi_0^{\mathrm{T}}\boldsymbol{\xi}). \quad (4)$$

Then the source location ξ_0 can be estimated by formulating the constrained optimization problem

$$\begin{aligned} &\underset{\xi_0,\gamma,\rho,\nu}{\text{minimize}}\ \|\boldsymbol{\Lambda}(\xi_0)\|_2, \\ &\text{subject to (2).} \end{aligned} \quad (5)$$

The minimization formulation (5) is non-convex, but it is amenable to SDR, i.e., the quadratic constraints in (2) can be relaxed by SDR, [15]. Thus, instead of (2), the following constraints are imposed:

$$\rho = \|\xi_1 - \xi_0\|_2,\ \begin{bmatrix} 1 & \rho \\ \rho & \nu \end{bmatrix} \succcurlyeq 0,\ \begin{bmatrix} \boldsymbol{I}_2 & \xi_0 \\ \xi_0^{\mathrm{T}} & \gamma \end{bmatrix} \succcurlyeq 0, \quad (6)$$

where $\boldsymbol{X} \succcurlyeq 0$ denotes positive semidefinite. With this, the localization problem reduces to an semidefinite programming (SDP), i.e., a convex minimization problem, solvable by standard convex optimization algorithms, [14],

$$\begin{aligned} &\underset{\xi_0,\gamma,\rho,\nu}{\text{minimize}}\ \|\boldsymbol{\Lambda}(\xi_0)\|_2, \\ &\text{subject to (6).} \end{aligned} \quad (7)$$

Note that formulation (7) is similar to that in [16], where a minimax formulation was used, i.e., $\|\boldsymbol{\Lambda}(\xi_0)\|_\infty = \max_{k=2,..,M} |\Lambda_k(\xi_0)|$ was minimized to estimate ξ_0, subject to the same constraints. However minimizing the ℓ_2-norm is equivalent to the LS formulation, which is known to be optimal given the Gaussian distribution of the TDOA estimates. Indeed, the simulation results in Sec. VI confirm that the ℓ_2-norm minimization can offer better accuracy than the minimax formulation.

IV. MXTM Method for Hyperbolic Localization

The aim of this section is to improve the localization accuracy over the traditional methods by incorporating the linearized version of the hyperbolic localization problem (traditionally solved by WLS), into a biased estimation framework discussed in [18, 20]. First, the linearized equations and the conventional solution WLS are presented. Then the biased estimation framework is introduced and the proposed integration of the hyperbolic localization problem is presented and discussed.

The non-linear equations (1) can be reorganized into a set of linear equations, by squaring and introducing an extra variable expressed as function of the source location, [7, 10-12]. Specifically, (1) can be rewritten

$$d_{k1} + \|\xi_1 - \xi_0\|_2 = \|\xi_k - \xi_0\|_2 + n_{k1} \quad (8)$$

By squaring both terms of the equality and introducing the new variable $\rho = \|\xi_1 - \xi_0\|_2$, (8) becomes

$$\begin{aligned} &(\xi_k - \xi_1)^{\mathrm{T}}\xi_0 + d_{k1}\rho = \\ &\qquad = \tfrac{1}{2}[(\xi_k - \xi_1)^{\mathrm{T}}(\xi_k + \xi_1) - d_{k1}^2] + e_k\,, \end{aligned} \quad (9)$$

where $e_k = n_{k1}(\|\xi_k - \xi_0\|_2 + n_{k1}/2)$ is the noise term. Denoting $\theta = [\xi_0^{\mathrm{T}}\ \rho]^{\mathrm{T}}$ and neglecting the second order noise term, (9) can be written in a matrix form,

$$\boldsymbol{G}\theta = \boldsymbol{h} + \boldsymbol{e}, \quad (10)$$

where $\boldsymbol{G} = \begin{bmatrix} \xi_2^{\mathrm{T}} - \xi_1^{\mathrm{T}} & d_{21} \\ \vdots & \vdots \\ \xi_M^{\mathrm{T}} - \xi_1^{\mathrm{T}} & d_{M1} \end{bmatrix}$, $\boldsymbol{h} = \begin{bmatrix} (\xi_2 - \xi_1)^{\mathrm{T}}(\xi_2 + \xi_1) - d_{21}^2 \\ \vdots \\ (\xi_M - \xi_1)^{\mathrm{T}}(\xi_M + \xi_1) - d_{M1}^2 \end{bmatrix}$, and $\boldsymbol{e} = [n_{21}\|\xi_2 - \xi_0\|_2, \dots, n_{M1}\|\xi_M - \xi_0\|_2\,]^{\mathrm{T}}$. Problem (10) is traditionally solved by minimization of a WLS objective function, as in [7]:

$$\hat{\theta} = \arg\min_{\theta}\,(\boldsymbol{G}\theta - \boldsymbol{h})^{\mathrm{T}}\boldsymbol{Q}_e^{-1}(\boldsymbol{G}\theta - \boldsymbol{h})\,, \quad (11)$$

where $\boldsymbol{Q}_e$ is an weighting matrix. Usually, the measurement noise n_{k1} is small enough compared to the distances $\|\xi_k - \xi_0\|_2$ such that $n_{k1}^2/2$ can be neglected and the noise term e_k can be modeled as a zero mean Gaussian random process with the covariance matrix $\boldsymbol{Q}_e = \boldsymbol{B}^T\boldsymbol{Q}_n\boldsymbol{B}$, where $\boldsymbol{B} = \mathrm{diag}\{\|\xi_2 - \xi_0\|_2, \dots, \|\xi_M - \xi_0\|_2\}$. Note that $\boldsymbol{B}$ depends on the unknown location ξ_0 and thus the WLS problem (11) is first solved with $\boldsymbol{Q}_e = \boldsymbol{Q}_n$ to obtain an estimate of $\boldsymbol{B}$ and then with $\boldsymbol{Q}_e = \widehat{\boldsymbol{B}}^T\boldsymbol{Q}_n\widehat{\boldsymbol{B}}$ to actually estimate θ. This method provides an approximation of the ML estimator for source location. However this approximation holds only when the errors in the TDOA estimates are small enough.

It was shown in [18, 21] that for linear systems such as (10) there exist biased estimates, which can provide better accuracy then the LS solution. The LS solution for linear systems is based on minimizing the ℓ_2-norm of the data error, $\widehat{\boldsymbol{h}} - \boldsymbol{h}$, where $\widehat{\boldsymbol{h}} = \boldsymbol{G}\hat{\theta}$, rather than minimizing the size of the estimation error, $\hat{\theta} - \theta$. To develop an estimation method that is based directly on the estimation error, an estimator $\hat{\theta}$ that minimizes the mean squared error (MSE) is desired. The MSE of an estimate $\hat{\theta}$ of θ is defined, [18],

$$\mathrm{MSE}(\hat{\theta}) = \mathbb{E}\left\{\|\hat{\theta} - \theta\|_2^2\right\} = \mathrm{var}\{\hat{\theta}\} + \|\mathrm{b}\{\hat{\theta}\}\|_2^2, \quad (12)$$

where $\mathrm{var}\{\hat{\theta}\} = \mathbb{E}\left\{\|\hat{\theta}\|_2^2\right\} - \left(\mathbb{E}\{\hat{\theta}\}\right)^2$ is the *variance* of the estimate and $\mathrm{b}\{\hat{\theta}\} = \mathbb{E}\{\hat{\theta}\} - \theta$ is the *bias* of the estimate. Since the bias generally depends on the unknown parameter θ, we cannot chose an estimator to directly minimize the MSE. A common approach is to restrict the estimator to be linear and unbiased and seek an estimator of this form that minimizes the variance $\mathrm{var}\{\hat{\theta}\}$. It is well known that the LS estimator minimizes the variance of the estimate $\hat{\theta}$ among all unbiased linear estimates. However this does not imply that the LS estimator has the smallest MSE. This motivates the approach of attempting to reduce the MSE by allowing some nonzero bias. Since the bias depends on the unknown θ, one solution is to exploit some a priori information on θ. For the localization problem, such information can consist in the limits of the surveillance area. With this, a biased estimation approach, denoted *minimax total MSE* (MXTM) in [18], can be employed to solve the hyperbolic localization problem. Assuming that the estimator is of form $\hat{\theta} = \boldsymbol{\Gamma} h$, for some $3 \times (M-1)$ matrix $\boldsymbol{\Gamma}$, and using it together with (10) in (12) it can be shown that the MSE of $\hat{\theta}$ is

$$\mathrm{MSE}(\boldsymbol{\Gamma}) = \mathrm{trace}(\boldsymbol{\Gamma}\boldsymbol{Q}_e\boldsymbol{\Gamma}^{\mathrm{T}}) + \theta^{\mathrm{T}}(\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G})^{\mathrm{T}}(\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G})\theta. \quad (13)$$

Exploiting the information that limiting the surveillance area places a bound on $\|\theta\|_2$, e.g., $\|\theta\|_2 < L$, the estimator can be expressed $\hat{\theta} = \hat{\boldsymbol{\Gamma}} h$, where

$$\hat{\boldsymbol{\Gamma}} = \arg\min_{\boldsymbol{\Gamma}} \max_{\|\boldsymbol{\Gamma} h\|_2 < L} \mathrm{MSE}(\boldsymbol{\Gamma}). \quad (14)$$

Problem (14) seeks to minimize the worst-case MSE across all possible estimators of θ, of the form $\boldsymbol{\Gamma} h$, with the ℓ_2-norm bounded by L. To solve the problem, the worst-case MSE is first determined. By algebraic manipulations, it can be shown that the worst-case MSE is $\mathrm{trace}(\boldsymbol{\Gamma}\boldsymbol{Q}_e\boldsymbol{\Gamma}^{\mathrm{T}}) + L^2\lambda_{\max}$, where $\lambda_{\max}$ is the maximum eigenvalue of $(\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G})^{\mathrm{T}}(\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G})$. It is known, [15], that $\lambda_{\max}$ can be obtained by

$$\text{minimize } \lambda, \quad (15)$$

$$\text{subject to} \quad \lambda\boldsymbol{I}_3 - (\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G})^{\mathrm{T}}(\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G}) \succcurlyeq 0, \quad (16)$$

By introducing (15) in (14), $\boldsymbol{\Gamma}$ can be estimated by

$$\text{minimize } \mathrm{trace}(\boldsymbol{\Gamma}\boldsymbol{Q}_e\boldsymbol{\Gamma}^{\mathrm{T}}) + L^2\lambda_{\max}, \quad (17)$$
$$\text{subject to (16)},$$

with variables $\boldsymbol{\Gamma}$ and λ. The constrained minimization (17) is a standard quadratic constrained quadratic problem, [15], that can be relaxed to an SDP,

$$\text{minimize } \beta, \quad (18)$$

$$\text{subject to} \quad \begin{bmatrix} \beta - L^2\lambda & \boldsymbol{g}^T \\ \boldsymbol{g} & \boldsymbol{I}_3 \end{bmatrix} \succcurlyeq 0, \quad (19)$$

$$\begin{bmatrix} \lambda\boldsymbol{I}_3 & (\boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G})^{\mathrm{T}} \\ \boldsymbol{I}_3 - \boldsymbol{\Gamma}\boldsymbol{G} & \boldsymbol{I}_3 \end{bmatrix} \succcurlyeq 0, \quad (20)$$

with variables $\beta, \boldsymbol{\Gamma}$, and λ, where $\boldsymbol{g} = \mathrm{vec}\{\boldsymbol{\Gamma}\boldsymbol{Q}_e^{1/2}\}$ denotes the vector obtained by stacking the columns of $\boldsymbol{\Gamma}\boldsymbol{Q}_e^{1/2}$.

Solving for (18)-(20) provides an estimate of $\theta = [\xi_0^{\mathrm{T}}\ \rho]^{\mathrm{T}}$. However, ρ was introduced into the hyperbolic localization problem as an intermediate variable that depends on ξ_0 through $\rho = \|\xi_1 - \xi_0\|_2$. This needs to be used into the minimization problem as an additional constraint. Introducing a new variable $\boldsymbol{\Theta} = \theta^{\mathrm{T}}\theta$, using $\hat{\theta} = \boldsymbol{\Gamma} h$, and employing SDR, it can be shown that the following two constraints can be introduced into the minimization (18)-(20) to account for the relation between ρ and ξ_0 and keep the problem convex at the same time:

$$\mathrm{trace}(\boldsymbol{P}\boldsymbol{\Theta}) + \xi_1^{\mathrm{T}}\xi_1 - 2\xi_1^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\Gamma} h = 0, \quad (21)$$

$$\begin{bmatrix} \boldsymbol{\Theta} & \boldsymbol{\Gamma} h \\ (\boldsymbol{\Gamma} h)^T & 1 \end{bmatrix} \succcurlyeq 0, \quad (22)$$

where $\boldsymbol{P} = \mathrm{diag}\{1, 1, -1\}$, and $\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Thus, the location estimate of ξ_0 is $\boldsymbol{R}\hat{\boldsymbol{\Gamma}} h$, where $\hat{\boldsymbol{\Gamma}}$ is obtained by solving the convex optimization problem

$$\text{minimize } \beta, \quad (23)$$
$$\text{subject to (19), (20), (21), and (22)},$$

with variables β, $\boldsymbol{\Gamma}$, and λ.

The simulation results in Sec VI show that the MXTM method offers location estimates of higher accuracy than the previous estimation approach, particularly for the case when the source is placed outside the convex hull of the sensors.

V. ℓ_1-NORM REGULARIZATION METHOD FOR HYPERBOLIC LOCALIZATION

The methods presented in Sec. III and Sec. IV offer high localization accuracy, as demonstrated by the simulation results shown in Sec. VI. However they solve a linear approximation of the hyperbolic localization problem and are suboptimal. In this section we propose a new approach that may offer even higher accuracy. This new approach exploits the source sparsity, i.e., the spatial sparsity. We propose to convert the localization problem to a sparse framework by solving the system $\mathbf{1} = \boldsymbol{A}\boldsymbol{z}$, where $\mathbf{1}$ is a unity vector whose length equals the number $M-1$ of TDOA estimates, τ_{k1}, for $k = 2, \dots M$, and $\boldsymbol{z}$ is a vector whose elements are associated with grid points, such that $\boldsymbol{z}_j \neq 0$ if a source is present at the grid. The elements $\boldsymbol{A}_{ij}$ are values of a function f chosen such that $f\left(\left|\tau_{k1}^{(j)} - \tau_{k1}\right|\right) = 1$, when the estimated TDOA associated with sensor k, τ_{k1}, equals the true TDOA, $\tau_{k1}^{(j)}$, calculated for sensor k and grid point j. Function f is chosen as a measure of the likelihood that the source is located at the grid point j. Thus, for grid points j for which $\tau_{k1}^{(j)} \neq \tau_{k1}$, function f takes values smaller than 1, such that $f\left(\left|\tau_{k1}^{(j)} - \tau_{k1}\right|\right)\Big|_{\tau_{k1}^{(j)} \neq \tau_{k1}}$ is monotonic decreasing. Estimation of $\boldsymbol{z}$ yields then the source location. Solving the system $\mathbf{1} = \boldsymbol{A}\boldsymbol{z}$ by traditional LS produces poor estimates since the number of unknowns, which equals the number of grid points, is usually much larger than the number equations, $M-1$, and thus matrix $\boldsymbol{A}$ is a fat matrix. The problem can be addressed by exploiting the source sparsity, which means that the size of the support of vector $\boldsymbol{z}$, or otherwise number of non-zero elements, is small relative to the length of $\boldsymbol{z}$.

Thus, the localization problem is formulated as an ℓ_1-regularization problem, i.e., the ℓ_1-norm is used to impose a sparsity constraint on vector $\boldsymbol{z}$, whose support indicates the source location:

$$\underset{\boldsymbol{z}}{\text{minimize}} \ \ \|\mathbf{1} - \boldsymbol{A}\boldsymbol{z}\|_2^2 + \lambda\|\boldsymbol{z}\|_1 , \quad (24)$$

where λ is a regularization parameter, balancing the fit of the solution $\boldsymbol{z}$ to the estimates τ_{k1} versus sparsity. This formulation is a convex optimization problem that can be efficiently solved by standard algorithms, [14].

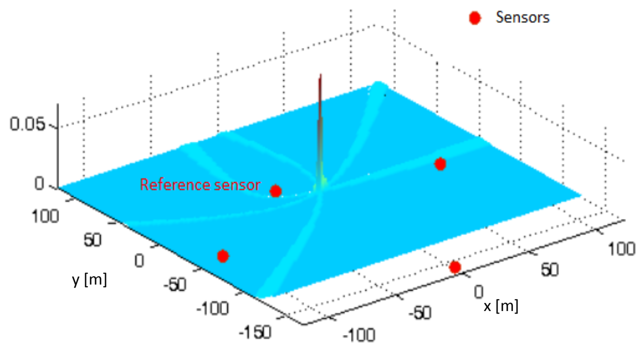

Fig. 2. Localization objective function obtained by ℓ_1-regularization.

Ideally, the sparsity of $\boldsymbol{z}$ is enforced by its ℓ_0-norm, i.e., the number of non-zero elements. However the minimization problem with the ℓ_0-norm constraint is a NP-hard non-convex optimization problem. By using the ℓ_1-norm as an approximation of the ℓ_0-norm, [22], the problem becomes convex. In Fig. 2 the estimated values of $\boldsymbol{z}$, acting as a localization objective function, are plotted at the corresponding space grid points, for a case with 4 sensors. The peak, which gives the location of the source, corresponds to the intersection of the hyperbolas associated with the 3 TDOA measurements available in this case.

A number of methods have been studied in the literature for automated choice of the regularization parameter λ (see [23] and the references therein). However, in practice, an optimal value of λ is difficult to select by any of these methods, and usually the choice of λ resorts to semi-empirical means, [24].

Formulating (24) with a denser sampled space has the potential of a higher resolution location estimate, but increases the complexity of the optimization algorithms. An iterative grid refinement approach is adopted to keep the complexity of the optimization algorithms in check. Initially, (24) is solved for the samples corresponding to a desired range of locations. A refined grid is obtained by taking a second set of samples focusing on an area that are indicated by the first iteration to include the source location. This corresponds to a higher sampling rate of the smaller area of interest. The transformation matrix $\boldsymbol{A}$ is also recalculated to match the refined sample support of the correlation sequences. With the refined grid, (24) is solved again and a new source location estimate, of higher resolution, is obtained. The grid refinement procedure can be repeated to improve the localization resolution. However, decreasing the grid spacing effects in high inter-column correlation in matrix $\boldsymbol{A}$. It is known, [25, 26], that as the inter-column correlation increases, the ℓ_1-regularization solution may become suboptimal, i.e., it does not coincides with the solution of the minimization with the ℓ_0-norm constraint. This sets an empirical lower bound on the localization resolution.

As demonstrated by the results in Sec. VI, the ℓ_1-regularization method has the potential of higher accuracy than the other two hyperbolic localization methods proposed. However, its performance depends on the choice of the regularization parameter. Also, a couple of iterations may be needed for grid refinement. Additionally, the localization resolution is limited by the grid spacing.

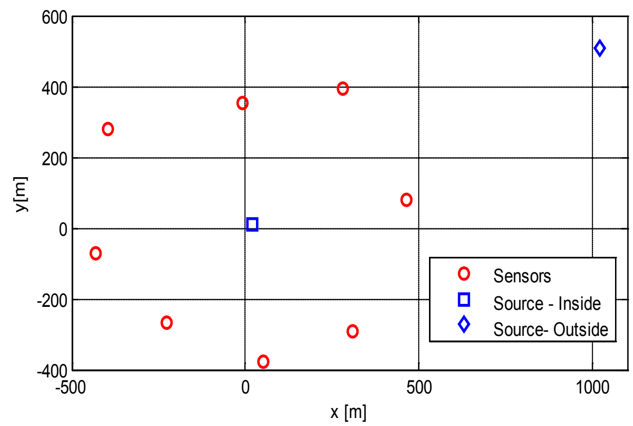

Fig. 3. Sensors layout. The source may be located inside or outside the sensors convex hull.

VI. Numerical Examples

Monte Carlo computer simulations were carried out for a number $M = 8$ sensors placed in the plane according to the layout in Fig. 3. Two cases were considered: one when the source is located inside the convex hull of the sensors and another one when the source is placed outside. The TDOA estimation errors were drawn from a zero-mean Gaussian distribution, with standard deviation σ_τ, where σ_τ was varied between 0 and 200 ns, i.e., the variance σ_n of the range differences varied between 0 and 60 m. For each value of σ_τ considered, 100 runs were performed. A zero-mean Gaussian function,

$$f\left(\left|\tau_{k1}^{(j)} - \tau_{k1}\right|\right) = \exp\left\{-\left|\tau_{k1}^{(j)} - \tau_{k1}\right|^2 / 2\sigma^2\right\}, \tag{25}$$

was used for simulations of (24), with $\sigma = 500$ ns. The plots in Fig. 4 and Fig. 5 show the root mean squared error (RMSE) of the methods proposed in this paper for a source placed with inside or outside of the convex hull of the sensors. The RMSE is plotted against the standard deviation of the TDOA estimation error.

The first remark is that all the three methods proposed in this paper outperform, for the cases simulated, the minimax approach from [16], known to be already more robust to errors in the TDOA estimates than conventional NLS and WLS methods for hyperbolic localization. The SDR method presented in Sec. III and MXTM presented in Sec. IV show similar accuracies when the source is placed inside the convex hull of the sensors, while MXTM performs better when the source is outside the convex hull. Both methods solve a linearized approximation of the hyperbolic localization problem. Finally, the ℓ_1-regularization outperforms for the simulated cases both the SDR and MXTM methods. In simulations, optimal choice of the regularization parameter was used. However, in practice a good choice of λ is difficult. A grid refinement procedure is needed if high resolution is desired, e.g., a number of five iterations were used in the simulations for a surveillance area of 1000 m by 1000 m, stopping at a grid resolution of 0.1 m.

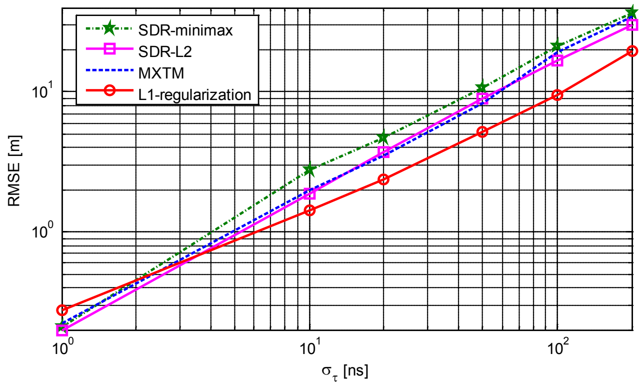

Fig. 4. Hyperbolic source localization for the case when the source is located inside the convex hull of the sensors.

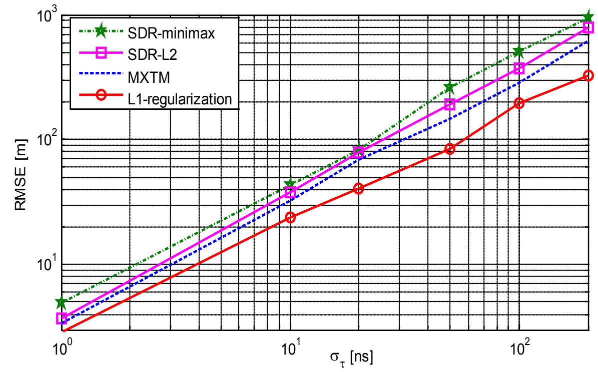

Fig. 5. Hyperbolic source localization for the case when the source is located outside the convex hull of the sensors.

VII. Conclusions

Three methods for hyperbolic localization were proposed to offer high accuracy at different computational costs. The first method is based on an SDR approach, the second method, MXTM, seeks a biased estimate through a linearized formulation of the localization problem, and the third method formulates the localization problem as an ℓ_1-regularization, by exploiting the sparsity of the source location. The proposed methods compare favorably with other existing methods, each of them having its own advantages. The SDR method has the advantage of simplicity and low computational cost. The MXTM may perform better than the SDR approach in some situations, but at the price of higher computational cost. The ℓ_1-regularization may outperform the first two methods, but is sensitive to the choice of the regularization parameter. Moreover, it may require a number of iterations to attain high, although limited, localization resolution.

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